Math 206A: Combinatorics of words (Prof. Igor Pak)

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1 Lecture 1: Introduction

1.1 First example

Consider an alphabet $a, b, c, \ldots$. There are two kinds of questions, existence and counting questions.

Let $A = \{0, 1\}$ and consider words in A^* with no w^2, for example with no instance of $(011)^2 = 011011$. Note that if the word starts with 0 the next letter has to be 1, the next letter has to be 0 and then no matter what the next letter is you are done. So all such words have $|w| \leq 3$.

Question 1.1. *If $A = \{0, 1, 2\}$, consider the words in A^* with no w^2. Is the size of the admissible words bounded?*

For example the word 0102012101201021 is admissible. In fact there are admissible words of arbitrary size.

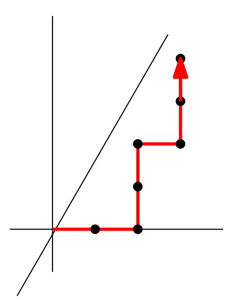
Theorem 1.2 (Thue 1906)**.** *There exists infinite words w in $\{0, 1, 2\}$ with no x^2.*

Theorem 1.3 (Thue 1906)**.** *There exists infinite words in $\{0, 1\}^*$ with no x^3.*

From these results we see that this subject is beyond enumeration.

1.2 Sturmian words

Consider the line $y = \alpha x$, for $\alpha \in \mathbb{Q}$, and paths starting at the origin with unit steps going right and up. We form words $w_\alpha \in \{a, b\}^*$ from each path by assigning the letter a if the step is the right and b if the step goes up. For example the path below corresponds to the word *aabbabb*.



Theorem 1.4. *Let $\gamma(n)$ the number of subwords of w_α of length n (so $w_\alpha \leq 2^n$). Then $\gamma(n) = n+1$ if $\alpha \notin \mathbb{Q}$.*

1.3 Gray codes

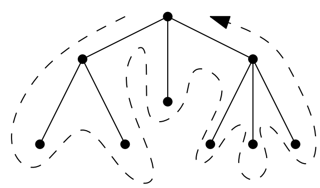
A Gray code is a sequence such that all k-subwords are distinct. A **De Bruijin** sequence is a cyclic sequence of length k from an alphabet A such that every possible sequence of length n in A appears as a sequence of consecutive characters exactly once. For example for $k = 1$ 01 words, for $k = 2$ 0011 works and for $k = 3$, 00011101 works.

There is a very nice formula for the number of such De Bruijin sequences.

1.4 Paths

Consider the number of paths starting at $(0, 0)$ with steps $(1, 1)$ and $(1, -1)$ staying above the line $y = 0$ and ending at $(2n, 0)$. The number of such paths are counted by the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Catalan numbers are standard in combinatorics and count many objects. For example, they also count rooted trees. To see the bijection, given a tree do a *tour* starting at the root around the tree[1]. As you traverse each edge record x if you are getting away from the root and y otherwise. For example for the tree below we get the word $xxyxyyxyxyxyxyy$.



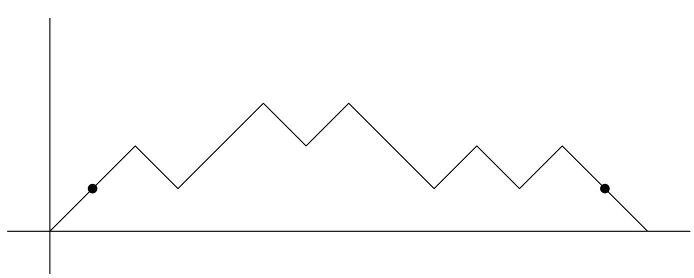
If you represent each path with a word, call these words *Dyck words*. Given two such words w_1 and w_2 then w_1w_2 is also a Dyck word. And if w is a Dyck word then xwy is also a Dyck word. This structure gives a recurrence, meaning that if a_n is the number of such words of length n then $A(x) = \sum_{n\geq 0} a_n t^n$ is algebraic. You can prove $A(x)$ satisfies a quadratic equation that you can solve and obtain that $a_n = \frac{1}{n+1}\binom{2n}{n}$.

We will see results relating properties of languages and properties of generating series (rational, algebraic, D-finite).

1.5 Combinatorics, asymptotics, and probability

Consider a random path that starts at the origin and you take unit paths to left or right with probability $\frac{1}{2}$. Let p_n be the probability of first return to the origin after $2n$ steps.

One can see that $p_n = \frac{2C_{n-1}}{2^{2n}} = \frac{C_n}{2^{2n-1}}$.



[1] If the root of the tree is the King and the nodes are the genealogy of who is next in the throne, the *tour* is the order of ascendence to the throne.

You can use this to compute the probability of return to the origin. An approximation of C_n is $\frac{c4^n}{\sqrt{\pi}n^{3/2}}$. Asymptotic combinatorics is interested in getting these approximations.

We will see some asymptotic conclusions using knowledge of the power series.

2 Lecture 2

2.1 Thue's theorems

Let $A = \{a_1, \ldots, a_k\}$ be the alphabet and A^* is the set of words $u = a_{i_1} a_{i_2} \cdots a_{i_\ell}$, the *length* of a word is denoted by $|w| = \ell$ and $\Sigma(A)$ denotes de the sets of infinite words in A.

Theorem 2.1 (Thue 1906). *For $A = \{a, b\}$, there exists w in $\Sigma(A)$ such that w is cube-free,* i.e. *w does not contain subword xxx for some $x \in A^*$.*

We will prove a stronger result which requires the following definition.

Definition 2.2. A word w is *strongly cube-free* if w has no subwords of the type xxc where c is the first letter of x. $\square$

The stronger result is the following.

Theorem 2.3 (Thue 1906). *For $A = \{a, b\}$, there exists w in $\Sigma(A)$ such that w is strongly cube-free.*

A *Thue word* is defined recursively as $w_1 = a$ and $w_{k+1} = w_k \cdot \overline{w_k}$. where $^{-}$ switches the characters of the word, *i.e.* $a \mapsto b$ and $b \mapsto a$. This gives the sequence of words

$$a, ab, abba, abbabaab, abbabaabbaababba, \ldots$$

Let $w = \lim_{k\to\infty} w_k$.

We show using a series of Lemmas that w is strongly cube-free.

We define a morphism h that takes $a \mapsto ab$ and $b \mapsto ba$.

Lemma 2.4. *h commutes with $^{-}$,* i.e. $\overline{h(w)} = h(\overline{w})$.

Proof. We check that the Lemma holds for the characters, $\overline{h(a)} = \overline{ab} = ba = h(\overline{a})$ and similarly for b. Then the result follows for words since $h(z_1 z_2 \cdots) = h(z_1)h(z_2)\cdots$ and $h(\overline{z_1 z_2} \cdots) = h(\overline{z_1})h(\overline{z_2}) \cdots = \overline{h(z_1)h(z_2)} \cdots$. $\square$

Lemma 2.5. *For $k \geq 1$ we have that $w_{k+1} = h(w_k)$.*

Example 2.6. *$w_2 = ab$, $h(w_2) = (ab)(ba) = abba = w_3$.*

Proof. We prove the Lemma by induction. For $k = 1$, $h(w_1) = h(a) = ab = w_2$. And for general k we have that $w_{k+1} = w_k \cdot \overline{w_k} = h(w_{k-1})\overline{h(w_{k-1})} = h(w_{k-1})h(\overline{w_{k-1}}) = h(w_{k-1}\overline{w_{k-1}}) = h(w_k)$. $\square$

Let $\varkappa(w, \ell) = \#(\ell\text{-subwords in } w) \leq 2^\ell$. Note that $\varkappa(w, 1) = 2$ and $\varkappa(w, 2) = 4$. However note that $\varkappa(w, 3) = 6$ (a^3 and b^3 do not occur in w; Lemma 2.7). So there is some stability.

By induction we can show the following lemma.

Lemma 2.7. *w has no a^3, b^3 or $(ab)^3$.*

Proposition 2.8. *$\varkappa(w, \ell)$ is computable.*

Also by induction we can show the following.

Lemma 2.9. *a^2 and b^2 can appear in w_k but only starting at even positions.*

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. We show by contradiction that w does not equal $(\cdots axxa \cdots)$ for some word x. Assume otherwise and take x to be minimal with such property. Observe that $|x|$ is even, otherwise if $|x|$ is odd then b^2 would land in an odd position which cannot happen by Lemma 2.9. It has to contain either an a^2 or a b^2. Say it has a b^2, so $x = \cdots bb \cdots$ then the previous letter to bb is a (b^3 is not allowed by Lemma 2.7), thus $x = \cdots abb \cdots$

But then w_{k-1} has $h^{-1}(x)h^{-1}(x)$.

□

Theorem 2.10. *Let w'_k be the word obtained from w_k by $aa \mapsto 1$ $ab \mapsto 2$ $ba \mapsto 3$ and $bb \mapsto 4$ where we run over 2-letter subwords.*

For example for $w = abbabaabbaab \cdots$ we get $2432212 \cdots$ then w'_k is square-free.

Theorem 2.11. *Let w''_k be the word obtained from w_k by $aa \mapsto 1$ $ab \mapsto 2$ $ba \mapsto 3$ and $bb \mapsto 1$ where we run over 2-letter subwords.*

For example for $w = abbabaabbaab \cdots$ we get $2132212 \cdots$ then w''_k is square-free.

2.2 Context

This is related to *Burnside's problem* that took 50 years to get a negative solution and 30 more years for a positive solution. Zelmanov received the Fields medal for his work on this problem. Let $B(n,k)$ be the free group G on n generators such that $g^k = 1$ for all g in G. $B(n,2) = \mathbb{Z}_2^n$. It is easy to see that $B(n,2)$ is abelian, since $(gh)^2 = 1$ and $g^2 = h^2 = 1$ then it follows that $hg = gh$. One can show that $B(n,3)$ is finite, it is more complicated to show that $B(n,4)$ is finite.

Question 2.12. *It is open to show that $B(n,5)$ is finite.*

We do not even know if $B(2,5)$ is finite. Another conjecture is the following.

Conjecture 2.13. *If $<\sigma, w>= A_n$, there exists a word of length $< 10^{10}$ such that word (σ, w) has order not divisible by 5.*

We are concerned with the version of this problem for semigroups.

3 Lecture 3:

We finish the proof of Thue's theorems.

We have proved that the word w is strongly-cube-free.

Theorem 3.1. *There exists a square-free word in $\{1,2,3\}^*$*

Recall that $w_{k+1} = w_k\overline{w_k} = h(w_k)$ where $h(a) = ab$ and $h(b) = ba$; $\overline{a} = b$ and $\overline{b} = a$. Define a map $\phi : [aa] \mapsto 1[ab] \mapsto 2, [ba] \mapsto 3, [bb] \mapsto 4$. And let $w' = \phi(w)$. So for example:

$$w = abbabaabbaababba \cdots$$
$$w' = 243231243123243 \cdots$$

Lemma 3.2. *The word w' is square-free in the alphabet $\{1,2,3,4\}$.*

Proof. Suppose w' contains a square then in w it comes from ww where w is an ab-word starting with a letter, say $z = a$, the letter after w is also z (since the pair of the last letter of w and z must appear again at the end of the second occurrence of w). Thus w contradicts Theorem 2.3. $\square$

Let $\psi : ab \mapsto 2, ba \mapsto 3, aa \mapsto 1, bb \mapsto 1$. Let $w'' = \psi(w)$.

$$\begin{aligned} w &= abbabaabbaababba\cdots \\ w'' &= 2132312131213123213\cdots \end{aligned}$$

Lemma 3.3. *$\psi(w) = w''$ is square-free in the alphabet $\{1, 2, 3\}$.*

Proof. The word w'' has the same information as w', since *baab* goes to 312 and *baab* goes to 243. So if there is a square in w'', there is a square in w'. $\square$

Remark 3.4. This construction has been rediscovered a number of times. Thue-Morse goes back to 1906 and 1930. But Prouhet had found a word in 1851 and Euwe in 1929. Thue was interested in Burnside's conjecture. $B(n, k) = F_n/(x^k = 1)$. Novikov and Novikov-Adjan (1950s, 1960s) showed there are > 667. And Ivanov showed that if k is even then $B(n, k) > 2^{48}$. $\square$

Theorem 3.5. *There exists infinite finitely generated groups with all elements of finite order.*

Proof. The first proof uses Golod-Shafarevich theorem. The second group is by Aleshin and Grigorchuk. $\square$

Remark 3.6. Max Euwe was a chess champion from Denmark. Currently if during a game of chess the same board is arrived at three times it is declared a draw. This is called *three-fold repetition.* In the history of this rule there was a predecessor used in a London tournament in 1883 said that if in the sequence of moves there are repeated moves repeated three times then there is a draw. Then it changed to six times for the first World Championship. Euwe discovered the Thue-Morse word, by showing that there are two or more possibilities, say just a and b then you can obtain infinite words that are cube-free. $\square$

3.1 Tarry-Escott Problem

This problem was stated in 1910 and 1912 but was solved by Prouhet in 1851. Wright (from Hardy-Wright discovered Prouhet's result).

Let $X, Y \subset \mathbb{Z}$ such that $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ is there a solution to the system

$$\left\{ \begin{array}{ll} x_1 + \cdots + x_n & = y_1 + \cdots + y_m \\ x_1^2 + \cdots + x_n^2 & = y_1^2 + \cdots + y_m^2 \\ \quad\vdots & \qquad\vdots \\ x_1^k + \cdots + x_m^k & = y_1^k + \cdots + y_m^k \end{array} \right\}$$

Example 3.7. *For $k = 5$ take $X = \{0, 5, 6, 16, 17, 22\}$ and $Y = \{1, 2, 10, 12, 20, 21\}$.*

Example 3.8 (Euler-Goldbach 1751)**.** *For $k = 2$, $X = \{a, b, c, a + b + c\}, Y = \{a + b, a + c, b + c\}$*

Theorem 3.9 (Prouhet)**.** *Let $w_k = abbabaab\cdots$ which has length 2^k, $X_k = \{1, 4, 6, 7, \ldots\}$ (positions of a in w_k) and $Y_k = \{2, 3, 5, 8, \ldots\}$ (positions of b in w_k). Then (X_k, Y_k) satisfies the Tarry-Scott problem up to the $k - 1$th power.*

Example 3.10. *$w_2 = abba$, $X_2 = \{1, 4\}$ and $Y_2 = \{2, 3\}$. For $w_3 = abbabaab$, $X_3 = \{1, 4, 6, 7\}$ and $Y_3 = \{2, 3, 5, 8\}$.*

4 Lecture 4

Let $X, Y \subset \mathbb{N}$ such that $X \cap Y = \varnothing$. We want (X, Y) to satisfy the system

$$\sum_{x \in X} x^\ell = \sum_{y \in Y} y^\ell, \quad \forall \ell = 0, \dots, k \tag{4.1}$$

Theorem 4.2 (Prouhet 1851)**.** *Let w be a Thue word, define $X_k, Y_k \subseteq [2^k]$ where X_k are the positions of a in w_k and Y_k are the positions of b in w_k then (X_k, Y_k) is TE_{k-1}.*

Lemma 4.3. *If (X, Y) is in TE_k then $(X + N, y + N)$ is TE_k*

Proof. For all $\ell = 0, \dots, k-1$ we have

$$\begin{aligned}\sum_{x \in X}(x+N)^\ell - \sum_{y \in Y}(y+N)^\ell &= \left(\sum_{x \in X} x^\ell + \sum_{y \in Y} y^\ell\right) + \ell N \left(\sum_{x \in X} x^{\ell-1} - \sum_{y \in Y} y^{\ell-1}\right) + \cdots \\ &= 0 + 0 + \cdots = 0.\end{aligned}$$

□

Lemma 4.4. *If (X, Y) is TE_k then $(2X, 2Y)$ is TE_k.*

Proof.

$$\sum_{x \in X}(2x)^\ell - \sum_{y \in Y}(2y)^\ell = 2^\ell \left(\sum_{x\ inX} x^\ell - \sum_{y \in Y} y^\ell\right) = 2^\ell \cdot 0 = 0.$$

□

Lemma 4.5. *If (X, Y) is TE_k then $(X \cup (Y+1), (X+1) \cup Y)$ is TE_{k+1}.*

Proof.

$$\begin{aligned}\sum_{x \in X} x^\ell - \sum_{x \in X}(x+1)^\ell - \sum_{y \in Y} y^\ell + \sum_{x \in X}(x+1)^\ell &= -\ell\left(\sum_{x \in X} x^{\ell-1} - \sum_{y \in Y} y^{\ell-1}\right) + \binom{\ell}{2}\left(\sum_{x \in X} x^{\ell-2} - \sum_{y \in Y} y^{\ell-2}\right) - \cdots \\ &= 0 - 0 + 0 - 0 \cdots = 0.\end{aligned}$$

□

Proof of Terry-Scott problem. Assume that (X_k, Y_k), a partition of $[1, 2, \dots, 2^k]$, is TE_{k-1} then $(X_k - 1, Y_k - 1)$ is also TE_{k-1} by Lemma 4.3, and so is $(2(X_k - 1), 2(Y_k - 1))$ by Lemma 4.4. The essence of Lemma 4.5 is what h does to w ($h(a) = ab, h(b) = ba$). □

But then by Lemma 4.5 $(X_{k+1} - 1, Y_{k+1} - 1)$ is TE_k and so by Lemma 4.3 (X_{k+1}, Y_{k+1}) is TE_k as desired.

4.1 Complexity of Thue word w

Recall that $\varkappa(n)$ is the number of subwords of length n in w.

Theorem 4.6. *$\varkappa(n) = O(n)$, moreover $\varkappa(n) \leq 4n$.*

Example 4.7. *This sequence is in [**?**][A005842].*

$$2, 4, 16, 10, 17, 16, 20, 22, 24, 28, 32, 36, \dots$$

$\varkappa(20) = 60, \varkappa(30) = 90, \varkappa(40) = 124, \varkappa(50) = 162.$

Lemma 4.8. *$\varkappa(2m+1) = 2\varkappa(m)$ and $\varkappa(2m) = \varkappa(m+1) + \varkappa(m)$.*

Theorem 4.9. *$\varkappa(n) = 3(n-1) + dist\{n-1,\ nearest\ 2^k\}$.*

Example 4.10. *For $n = 49$ we have $\varkappa(49) = 3 \cdot 48 + 16 = 160$ since 48 is in the middle of the powers 32 and 64..*

This can be proved by induction.

Sublemma 4.11. *Let u be a word of length ≥ 4. If $u = (ab \dots u \cdots u \cdots)$ then us have positions of same parity.*

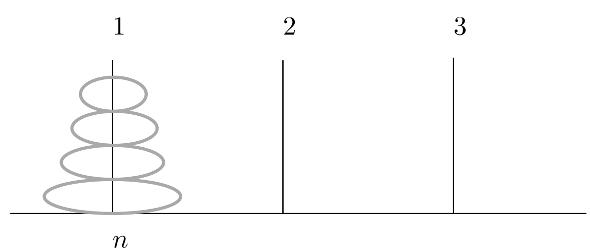
Next we show that $\varkappa(2m+1) = 2 \cdot \varkappa(m+1)$. If we count the number of subwords u of odd length that start in odd position equals $\varkappa(m+1)$.

$w_k = (abba \dots U \dots)$ then $w_{k-1} = (ab \dots h^{-1}(u)c \dots)$ and $h^{-1}(u)$ has length $m+1$.

Similarly, if we count subwords u of odd length that start in even position equals $\varkappa(m+1)$.

A similar argument proves $\varkappa(2m) = \varkappa(m+1) + \varkappa(m)$.

5 Lecture 5: Tower of Hanoi



Consider the **Tower of Hanoi** game with n disks.

Theorem 5.1. *The minimal number of steps in the Tower of Hanoi game with n discs is $2^n - 1$.*

Proof. Let $\gamma(n)$ be the number of steps as in the statement. We prove it by induction. We check that $\gamma(1) = 1$. Now $\gamma(n) \leq \gamma(n-1) + 1 + \gamma(n-1)$ by construction. This shows inductively that $f(n) \leq 2^n - 1$.

For the lower bound we also show $\gamma(1) = 1$. Note that for the biggest disc to move from peg 1 to peg 2 we also need $\gamma(n) \geq \gamma(n-1) + 1 + \gamma(n-1)$. Therefore $\gamma(n) \geq 2^n - 1$.

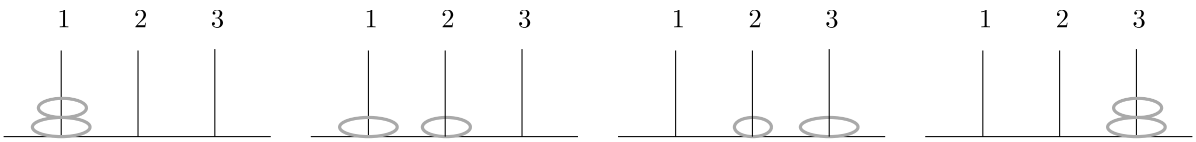
Thus $\gamma(n) = 2^n - 1$. ☐

5.1 Tower of Hanoi word

Let $a : 1 \to 2$, $b : 2 \to 3$, $c : 3 \to 1$; and $\overline{a} : 2 \to 1, \overline{b} : 3 \to 2, \overline{c} : 1 \to 3$. These six letters completely encode the operations of the tower of Hanoi.

Let H_n be the word of tower of Hanoi with n discs from peg 1 to peg 2 if n is odd and from peg 1 to peg 3 if n is even.

Example 5.2. *$H_1 = a$, $H_2 = a\overline{c}b$, $H_3 = a\overline{c}bac\overline{b}a$, $H_4 = a\overline{c}bac\overline{b}a\overline{c}b\overline{a}cba\overline{c}b$.*



Remark 5.3. There exists $H = \lim_{n\to\infty} H_n$, since H_{n-1} is a prefix of H_n. $\square$

Question 5.4. *What is H?*

Let $\Psi : \{a, \overline{a}\} \mapsto a, \{b, \overline{b}\} \mapsto b, \{c, \overline{c}\} \mapsto c$, and $\varphi : \{a, b, c\} \mapsto 0, \{\overline{a}, \overline{b}, \overline{c}\} \mapsto 1$. Let $G_n = \Psi(H_n)$, $B_n = \varphi(H_n)$ and $G = \lim_{n\to\infty} G_n$ and $B = \lim_{n\to\infty} B_n$. Note that determining H is equivalent to (G, B).

Example 5.5. *$G = acbacbacbcbacb \cdots = (acb)^\infty$. $B = 010001010100010 \cdots$. Recall the Thue word $a\underline{bb}ab\underline{aa}\,\underline{bb}aba\underline{bb}a \cdots$. The squares in the Thue word correspond to the ones in G.*

Theorem 5.6. *$G = (acb)^\infty$ and B is a sequence of 1s at the squares of the Thue word W.*

Proof. Let $\sigma : a \mapsto b \mapsto c \mapsto a$ and $\overline{a} \mapsto \overline{b} \mapsto \overline{c} \mapsto \overline{a}$.

Lemma 5.7. $H_n = \begin{cases} H_{n-1}\overline{c}\sigma(H_{n-1}) & \text{if } n \text{ even,} \\ H_{n-1}a\sigma^2(H_{n-1}) & \text{if } n \text{ odd.} \end{cases}$.

$\square$

Next we look at the case of the game of Tower of Hanoi with more then three pegs. Let n be the number of discs and k be the number of pegs. Let $h = k - 2$ and s be the unique integer such that $\binom{h+s-1}{h} < n \le \binom{h+s}{h}$, then

Conjecture 5.8. *The number $H_k(n)$ of steps in the Hanoi tower game with m pegs and n discs equals $a_k(n)$ where*

$$a_k(n) = 2^s \left(n - \binom{h+s-1}{h} \right) + \sum_{t=0}^{s-1} 2^t \binom{h+t-1}{h},$$

where h and s as defined above.

Theorem 5.9 (Hinz, 1988)**.** *The number $H_k(n)$ of moves in the game of towers of Hanoi with n discs and m pegs is $H_k(n) \le a_k(n)$.*

Theorem 5.10 (M. Szegedy 1998)**.** *The $H_k(n) = 2^{\Theta(n^{1/(k-2)})}$, and $b_k(n) \ge a_k(n)^{C_k}$ for some constant C_k.*

A growth of the form $e^{\sqrt{n}}$ is called an **intermediate growth**.

6 Lecture 6:

Lucas came up with Hanoi towers in 1883.

Let TH_k be the game of tower of Hanoi with k pegs.

Theorem 6.1. *Let $H_k(n)$ be the minimum number of moves for TH_k. Then $TH_k(n) \leq a_k(n) = 2^s \left(n - \binom{h+s-1}{h}\right) + \sum_{t=0}^{s-1} 2^t \binom{h+t-1}{h}$.*

Stewart posed this problem and gave the bound in the American Mathematical Monthly in 1941, Frame (the same Frame as in the hook-length formula) also found the bound with essentially the same proof.

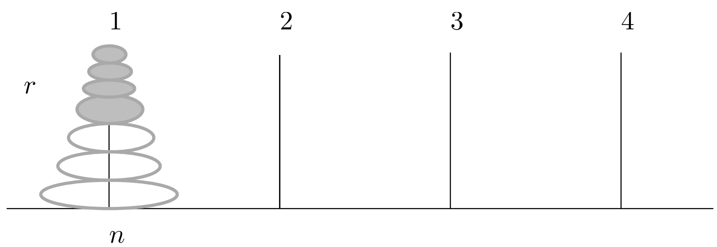
Theorem 6.2 (M. Szegedy 1998)**.** $H_k(n) = 2^{\Theta_k(n^{1/(k-2)})}$.

Theorem 6.3 (Chen, Shen, 2004)**.** $H_k(n) = 2^{(1+o(1))(k-2)!-n)^{1/(k-2)}}$*. Moreover they show* $H_k(n) = n^{o(1)} \cdot 2^{[(k-2)!n]^{1/(k-2)}}$.

6.1 Frame-Stewart Algorithm (k=4)

Let $f(n)$ be the number of moves with n discs. Take $1 \leq r \leq n$ and move r discs to peg 3. This leaves $n-r$ discs to move in three pegs, which reduces to $H_2(n-r)$. Then we move the r pegs to where the $n-r$ pegs are now. Thus

$$f(n) = f(r) + (2^{n-r} + 1) + f(r).$$



For which r do we get the best recurrence? We claim that this happens when r is large, around $2^{\sqrt{n}}$. So $n - r = \Theta(\sqrt{n})$. Thus $f(n) \leq n^c \cdot 2^{\sqrt{2n}}$. You want to minimize the time you just use three pegs.

Let $n = \binom{\ell}{2}$. We show by induction on ℓ that $f(n) \leq (\ell + c) \cdot 2^\ell$ for some $c \geq 0$. By induction hypothesis $f(\ell(\ell-1)/2) \leq (\ell+c)2^\ell$ then

$$f(\ell(\ell+1)/2) \leq 2 \cdot f(\ell(\ell-1)/2) + (2^\ell - 1) \leq 2(\ell + c) \cdot 2^\ell + 2^\ell \leq (\ell + 1 + c)2^{\ell+1}.$$

We also need to check the base case and we are done.

For the lower bound instead of counting $f(n)$, we count $g(n)$ defined as the minimal number of steps to move all discs. For example in the case of three pegs, when you get to move the largest disc you are done. The argument shows that $f(n)$ and $g(n)$ have the same asymptotics up to a polynomial factor.

Lemma 6.4. $g(n) \geq 2 \cdot \min\{g(n-8m), 2^{m-2} - 1\}$ *for every m.*

Sketch. From the definition of $g(n)$ it is difficult to nail the configuration for an induction. So we consider $g'(n)$ to be the number of steps to move all discs from any position. Consider the largest $8m$ discs, where $m \leq n/8$. There is at least one peg with $2m$ discs. Call the first L discs *large* and the m ones in the bottom *extra large*. The rest of the discs are called *small*. It takes $2^{m-2}-1$ moves before we touch the largest of the large discs on three pegs. And there $n-8m$ small discs which takes $g(n-8m)$ steps to move them. We get $g(n-8m)+2^{m-2}+1$ steps and this $\geq 2\min\{g(n-8m), 2^{m-2}-1\}$. From here we optimize to produce the lower bound. $\square$

D. Knuth has a quote when he was studying if this algorithm was optimal. "This is a difficult problem and no one can do it".

The *Sierpinski gasket* with three pegs the behavior is exponential with four pegs there is a limiting behavior but it is much more complicated.

7 Lecture 7

7.1 Sturmian words

Let $A = \{a, b\}$ and $w \in \Sigma(A)$ be an infinite word. $\varkappa(w, n)$ is the number of distinct subwords of length n in w. If $\varkappa(w, 1) = 1$ then $w = a^\infty$ or b^∞.

Assume $\varkappa(w, 1) = 2$ from now on.

Theorem 7.1. *Either $\varkappa(w, n) \geq n+1$ or $\varkappa(w, n) \leq C$ for some constant C.*

Moreover w is *eventually periodic* if and only if $\varkappa(w, n) \leq C$ for some constant C, this is equivalent to $w = u \cdot v^\infty$ for some $u, v \in A^*$.

Lemma 7.2. *If $\varkappa(w, n) = \varkappa(w, n+1)$ then w is eventually periodic.*

Proof. Suppose w is not eventually periodic, then $\varkappa(w, 1) = 2, \varkappa(w, 3) \geq 2, \varkappa(w, 3) \geq 4, \ldots$ $\varkappa(w, n) \geq n+1$ by induction.

Suppose $\varkappa(w, n) = \varkappa(w, n+1)$, then for every $u \in w$, $|u| = n$, u is a subword of w. Either ua or $ub \in w$ but not both.

There is at most 2^n words of length n, so somve v is repeated, from this point on w is periodic. $\square$

Definition 7.3. Let $A = \{0, 1\}$. The word v in $\Sigma(A)$ is *Sturmian*, if for all n, we have $\varkappa(w, n) = n+1$. $\square$

Definition 7.4. The *Fibonacci word* is defined as the limit of the following words: $F_0 = 0, F_1 = 01$ and $F_{n+1} = F_n F_{n-1}$. Then $w = \lim_{n\to\infty} F_n \in \Sigma(A)$. $\square$

$$F_0 = 0, F_1 = 01, F_2 = 010, F_3 = 01001, F_4 = 01001010, F_5 = 0100101001001.$$

Proposition 7.5. *$\varkappa(w, 1) = 2$, $\varkappa(w, 2) = 3$ (all F_n start with 0), $\varkappa(w, 3) = 4$.*

Proof. $\varkappa(w, 2) = 3$ since there is no 11. $\varkappa(w, 3) = 4$, the subwords are $001, 010, 100, 101$. This follows since $111, 110, 011$ are not present since we exclude 11. To see that 000 is also excluded, observe that everything ends either 01 or 10. $\square$

The main result on the Fibonacci word w is that it is Sturmian.

Theorem 7.6. *The Fibonacci word w is Sturmian.*

To prove this we need the following Lemma

Lemma 7.7. *$F_n = \varphi^n(0)$ where $\varphi(0) = 01$ and $\varphi(1) = 0$.*

Remark 7.8. Recall that the Thue morphism was $\varphi(0) = 01$ and $\varphi(1) = 10$. □

The Lemma is is equivalent to $F_{n+1} = \varphi(F_n)$. Then $w = \varphi(w)$.

Proof. We use strong induction on n. $F_1 = \varphi(F_0) = \varphi(0) = 01$. Now

$$\varphi(F_n) = \varphi(F_{n-1}F_{n-2}) = \varphi(F_{n-1})\varphi(F_{n-2}) = F_nF_{n-1} = F_{n+1}.$$

as desired. □

Lemma 7.9. *For all words u either $0u0$ or $1u1$ is not in w.*

Proof. For $u = \varnothing$ the result follows since 11 is not in w.

For $u = 1$ or $u = 0$, the result follows since 111 and 000 are not in w.

By contradiction suppose $0u0, 1u1$ are in w then $u = 0v0$ for some v since we cannot have $1v0$ or $1v1$, then $00v00, 10v01$ are in w. But $w = \varphi(w)$ so there exists z in w such that $\varphi(z) = 0v$. Therefore, $00v00 = \varphi(1z1)$ and $010v01 = \varphi(0z0)$. □

Corollary 7.10. $\varkappa(w, n+1) \leq \varkappa(w, n) + 1$.

Proof. There exists u, v in w such that $|u| = |v| = n$ and both $u0, u1, v0, v1$ are in w. Then $u = \underline{\ldots 0x \ldots}$ which implies $0x0 \in w$ and $v = \underline{\ldots 1x \ldots}$ which implies $1x1 \in w$. This contradicts Lemma 7.9. □

Lemma 7.11. *w is not eventually periodic.*

Proof. Let $\widetilde{F_n}$ denote F_n reversed. We claim that $\widetilde{F_n}0$ and $\widetilde{F_n}1$ are in w. There exist words of arbitrary length that can be extended. □

8 Lecture 8

8.1 Fibonacci word

A word $w \in \Sigma(0, 1)$ is *Sturmian* if $\varkappa(w, n) = n + 1$ for all $n \geq 1$.

Theorem 8.1. *$F_0 = 0, F_1 = 01, F_{n+1} = F_nF_{n-1}$, $w = \lim_{n\to\infty} F_n$ is called the w-Fibonacci word. Then w is Sturmian.*

Lemma 8.2. *For any w, if $\varkappa(w, n) = \varkappa(w, n+1)$ for some n then w is eventually periodic.*

Lemma 8.3. *For the w-Fibonacci word $\varkappa(w, n+1) \leq \varkappa(w, n) + 1$.*

Both Lemmas imply that the Fibonacci word w is eventually periodic or Sturmian.

Lemma 8.4. *w is not eventually periodic.*

Let $h(w)$ be the number of ones in w, and let $\pi(v) = \frac{h(v)}{|v|}$. If w is the Fibonacci word w then say $\pi(w) = \lim_{n\to\infty} \pi(w_n)$, where w_n is the n-prefix of w.

Note that if w is eventually periodic then $\pi(w) \in \mathbb{Q}$.

Lemma 8.5. *For the Fibonacci word w, $\pi(w) \notin \mathbb{Q}$.*

$|F_n| = f_{n+1}$ where $(f_0, f_1, \ldots)$ are the Fibonacci numbers. And $h(F_n) = f_{n-1}$, so $\pi(F_n) = \frac{f_{n-1}}{f_{n+1}} \to \frac{1}{\phi} \notin \mathbb{Q}$ where $\phi = \frac{\sqrt{5}+1}{2}$.

$$0, 01, 010, 01001, 01001010, 0100101001001.$$

Lemma 8.6. *We have that* $F_{n+2} = (F_{n-3} \cdots F_1 F_0)\widetilde{F_n}\widetilde{F_n} t_n$ *where* $t_n = \begin{cases} 01 & \textit{if } n \textit{ odd,} \\ 10 & \textit{if } n \textit{ even} \end{cases}$*, and where* $\widetilde{w}$ *is the reverse of* w*.*

Proof. Proof by induction (exercise). □

Corollary 8.7. $\kappa(w, n) < \kappa(w, n+1)$

Since $\widetilde{F_n} = ...\widetilde{F_{n-1}}$ then $\widetilde{F_n}0$ and $\widetilde{F_n}1$ are subwords of w. And for every suffix u of $\widetilde{F_n}$ satisfies $u0$ and $u1$ are subwords of w. So every subword extends from length n to $n+1$ and there words that extend into more than one word.

8.2 w-recurrent words

Definition 8.8. A word w is *recurrent* if every subword of w occurs infinitely many times. □

Remark 8.9. The Fibonacci word is recurrent, since for a subword u of w there exists an index n such that u is a subword of F_n, then u occurs at least i times inf F_{n+i}. □

Proposition 8.10. *If* w *is Sturmian then* w *is recurrent.*

Proof. Suppose there exists a subword u, with $n = |u|$ such that u occurs finitely many times. Let w' be the suffix of w where there re no more instances of u. But $\varkappa(w', n) \leq \varkappa(w, n) - 1 \leq n+1 = n$. Thus w' is eventually periodic and so w is eventually periodic. □

Definition 8.11. $X \subseteq A^*$, $X = \bigcup_{n=0}^{\infty} X^n$ is *factorial* if for all $x \in X$, Y is a subword of x then $y \in X$. □

Definition 8.12. $X, Y \in A^n$, meaning $|X| = |Y| = n$, then $\delta(X, Y) = |h(X) - h(Y)|$. X is *balanced* if for all $U, V \in X$ such that $|U| = |V|$ then $\delta(U, V) \leq 1$. □

The goal is to show the following result.

Theorem 8.13. *w is Sturmian if and only if w is balanced and not eventually periodic.*

Proposition 8.14. *If X is factorial and balanced then* $|X^n| \leq n+1$*.*

Proof. We show this by contradiction. For $n \geq 3$ take the smallest counterexample. Either $|X^{n-1}| \leq n$ or $|X^n| \geq n+2$. By minimality of u there exist $u, v \in X^{n-1}$, then $u0, u1, v0, v1 \in X^n$. We look at the longest suffix x that $u0$ and $v1$ have in common. Then $0x0, 1x1 \in X^k$ and this is a contradiction since it implies that X is not balanced. □

9 Lecture 9

Theorem 9.1. *The following are equivalent, for $w \in \Sigma(\{0,1\})$:*

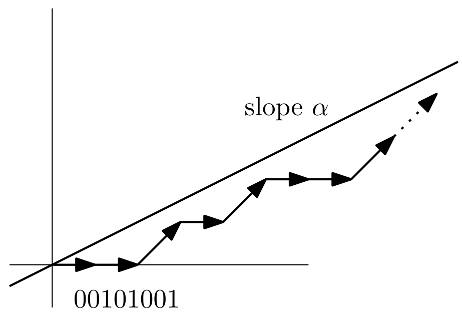
(1) w is Sturmian,

(2) w is balanced and aperiodic,

(3) w is mechanical *with irrational slope.*

Corollary 9.2. *Set of Sturmian words is uncountable.*

Definition 9.3. A word w is *mechanical* with slope α if it comes from a path with steps $(1,0)$ and $(1,1)$ in the positive quadrant below the line $y = \alpha x$. □



For example the Fibonacci word w has slope $\alpha = 1/\phi^2$.

Remark 9.4. If $\alpha \notin \mathbb{Q}$ then w not eventually periodic. □

To prove (3) → (2) from Theorem 9.1, w is balanced.

Remark 9.5. Recall the geometry of continued fractions. If

$$\alpha = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots}}}}$$

If p_n is the truncation of the continued fraction up to the nth term then $\lim_{n\to\infty} \frac{p_n}{a_n}$. Let $z_n = (a_n, p_n)$ and plot these points. The convex hull of these points is below the line $y = \alpha x$.

Claim 9.6. *A prefix x in w of length q_n has $h(x) = p_n$.*

□

Remark 9.7. Sturmian words give a combinatorial model to study irrational numbers. □

The steps of the proof of Theorem 9.1 are $(1) \to (2), (2) \to (3), (3) \to (2), (2) \to (1)$.

9.1 Proof $(2) \to (1)$ of Theorem 9.1

We know that w is aperiodic then $\varkappa(u,h) \geq n+1$. If w is balanced then $\varkappa(u,n) \leq n+1$. This implies that w is Sturmian.

9.2 Proof $(1) \to (2)$ of Theorem 9.1

Lemma 9.8. *Given a factorial word X, X is unbalanced if and only if there exists $w = \widetilde{w}$ palindrome such that $0w0, 1w1 \in X$.*

Given the lemma, if we have a Sturmian word w, we show it is balanced by contradiction. If it is unbalanced then it has a palindrome $0w0$ and $1w1$ for w palindromic. This then implies that $\varkappa(w, n) \neq n + 1$ for some n.

9.3 Proof $(2) \to (3)$ of Theorem 9.1

Let $\pi(x) = \frac{h(x)}{|x|}$ is the slope, then $\lim_{|x|\to\infty} \pi(x)$ exists, call this $\pi(w)$. This is α.

Lemma 9.9. *Given a factorial word X, X is balanced if and only if for all $u, v \in X$ then $|\pi(u) - \pi(v)| < \frac{1}{|u|} + \frac{1}{|v|}$.*

Note that we do not impose u and v have the same length.

10 Lecture 10

We finish Sturmian words. We continue the proof of Theorem 9.1. We do $(1) \to (2), (3) \to (2), (2) \to (1)$.

The use of *mechanical* words are words with steps $(1, 0)$ and steps $(1, 1)$ below the line $y = \alpha x + \beta$ where $0 \leq \beta < 1$ and $\alpha < 1$.

Proposition 10.1. *X is factorial, then X is balanced if and only if for all $u, v \in X$, $|x(u) - x(v)| < \frac{1}{|u|} + \frac{1}{|v|}$.*

Proof. ($\Leftarrow$) Take $|u| = |v| = n$ then $|\pi(u) - \pi(v)| < \frac{2}{n}$. Since the LHS of the inequality is an integer then $|\pi(u) - \pi(v)| \leq 1$ so w is balanced.

($\Rightarrow$) If x is balanced then $|u| > |v|$, $u = zt$ with $|z| = |v|$. By induction $|\pi(t) - \pi(u)| < \frac{1}{|t|} + \frac{1}{|v|}$, so $|h(z) - h(v)| \leq 1$ since x is factorial and balanced so $|\pi(z) - \pi(v)| < \frac{1}{|v|}$.

$$\begin{aligned}
\pi(u) - \pi(v) &= \frac{|z|}{|u|}\pi(z) + \frac{|t|}{|u|}\pi(t) - \pi(v), \\
&= \frac{|z|}{|u|}(\pi(z) - \pi(v)) + \frac{|t|}{|u|}(\pi(t) - \pi(v))
\end{aligned}$$

Thus

$$|\pi(u) - \pi(v)| < \frac{1}{|u|} + \frac{|t|}{|u|}\left(\frac{1}{|v|} + \frac{1}{|t|}\right) = \frac{1}{|u|} + \frac{1}{|v|}.$$

□

This proposition implies that for the nth prefix w_n of w, the sequence $(\pi(w_n))$ is a Cauchy sequence. Does it converges.

Corollary 10.2. *There exists a limit $\lim \pi(w_n) = \alpha$ where w_n is the nth prefix of w for all infinite balanced words.*

We are now able to prove (3) $\to$ (2) in Theorem 9.1.

$$\alpha_{n-2} < \lfloor \alpha n \rfloor < h(w_n) \leq \lfloor \alpha n \rfloor < \alpha n$$

$0 \leq |h(u) - h(v)| < 2$ since α is irrational.

Remark 10.3. Generalizing Sturmian words to higher dimensions is of interest. $\square$

11 Lecture 11

11.1 Quasideterminants

What is the noncommutative notion of determinants like $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$, that preserves $\det(A \cdot B) = \det(A)\det(B)$ and Cramer's rule $(A^{-1})_{ij} = (-1)^{i+j}\frac{\det(A^{ij})}{\det(A)}$?

Our entries will be in a division ring R.

Let $A = (a_{ij})_{n\times n}$ where $a_{ij} \in R$, denote by $|A|_{ij} = (A^{-1})_{ij}$. This is called the (i, j) *quasideterminant* of A.

Proposition 11.1. *Suppose R is a commutative ring, $A = (a_{ij})_{n,n}$, $a_{ij} \in R$. Then A is invertible over $\mathcal{F}(R)$ the ring of formal variables.*

Proof. For $n = 1$, $(a_{11})^{-1} = a_{11}^{-1}$.

For $n = 2$, $A = \begin{bmatrix} A^{n,n} & c \\ B & a_{nn} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, define $A^{-1} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$, where

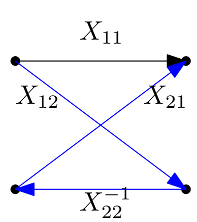
$$\begin{aligned}
Y_{11} &= (X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1}, \\
Y_{12} &= -X_{11}^{-1}X_{12}(X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1} \\
Y_{21} &= -X_{22}^{-1}X_{21}(X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1} \\
Y_{22} &= (X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1},
\end{aligned}$$

where X_{11}^{-1} is well defined by induction and $X_{22}^{-1} = a_{nn}^{-1}$.

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

For example the $(1, 1)$ entry is

$$\begin{aligned}
X_{11}Y_{11} + X_{12}Y_{21} &= X_{11} \cdot (X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1} + X_{12}X_{22}^{-1}X_{21}(X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1} \\
&= (X_{11} - X_{12}X_{22}^{-1}X_{21})(X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1} = 1.
\end{aligned}$$



$$Y_{11} = (X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1}$$

Once we figure out what happens for $n = 2$, we know what happens in general. In the case $n = 2$ above we can treat X_{11} as an $(n-1)\times(n-1)$ matrix. $\square$

Remark 11.2. We think of each term as an infinite product.

$$(x_{11} - x_{12}x_{22}^{-1}x_{21})^{-1} = [x_{11}(1 - x_{11}^{-1}x_{12}x_{22}^{-1}x_{21})]^{-1} = \frac{1}{1 - x_{11}^{-1}x_{12}x_{22}^{-1}x_{21}}x_{11}^{-1}.$$

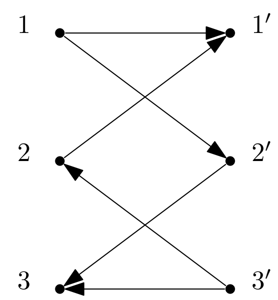
$\square$

Definition 11.3 (Quasideterminant)**.** Let $|A|_{ij} = (y_{ji})^{-1}$ where $(A^{-1})_{n,n} = (y_{ij})_{n,n}$. This is the (i,j) *quasideterminant* of A. $\square$

Example 11.4 ($n = 2$)**.** $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, *then* $|A|_{11} = a_{11} - a_{12}a_{22}^{-1}a_{21}$, $|A|_{12} = a_{12} - a_{11}a_{21}^{-1}a_{22}$, $|A|_{22} = a_{22} - a_{21}a_{11}^{-1}a_{12}$.

Example 11.5 ($n = 3$)**.**

$$\begin{aligned}|A|_{11} &= \begin{bmatrix} \underline{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ &= a_{11} - a_{12}(a_{22} - a_{23}a_{33}^{-1}a_{32})^{-1}a_{21} - a_{12}(a_{32} - a_{33}a_{23}^{-1}a_{22})^{-1}a_{31} - \\ &- a_{13}(a_{23} - a_{22}a_{32}^{-1}a_{33})^{-1}a_{21} - a_{13}(a_{33} - a_{32}a_{22}^{-1}a_{23})^{-1}a_{31}.\end{aligned}$$



We claim that the terms of the quasideterminant are words.

$$|A|_{11} = \sum_{w:1\to 1',\text{ no intermediate }1,1'} \prod_{a_{ij}\in w} a_{ij}.$$

Theorem 11.6. $|A|_{k,\ell} = \pm\sum_{w:k\to\ell',\ \textit{no intermediate}\ k,\ell'} \prod_{a_{ij}\in w} a_{i,j}^{\pm 1}$.

11.2 Cartier-Foata determinants

$a_{ij}a_{k\ell} = a_{k\ell}a_{ij}$ for $i \neq k$ then define the Cartier-Foata determinant as

$$\det_{CF}(A) = \sum_{\sigma\in S_n} (-1)^{sign(\sigma)} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}.$$

Theorem 11.7. $(-1)^{i+j}|A|_{ij} = \det_{CF}(A^{ij})^{-1}\det_{CF}(A)$.

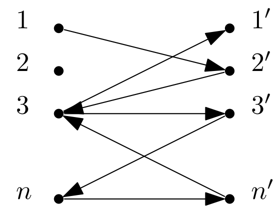
12 Lecture 12

Recall the connection between quasideterminants and words.

Theorem 12.1.

$$|A|_{11} = \sum_{w:1\to 1',\ no\ intermediate\ 1,1'} (-1)^{\frac{\ell(w)-1}{2}} wt(w).$$

For example the weight of the word φ is $a_{12}a_{32}^{-1}a_{33}a_{13}^{-1}a_{nn}a_{3n}^{-1}a_{31}$. The words have no instance of xx^{-1}.



Lemma 12.2 (Homological relations)**.** *For rows* $-|A|_{ij}|A^{i\ell}|_{sj} = |A|_{i\ell}|A^{ij}|_{s\ell}^{-1}$ *for* $s \neq i$ *and* $\ell \neq j$. *For columns* $-|A^{kj}|_{it}^{-1}|A|_{ij} = |A^{ij}|_{kt}^{-1}|A|_{kj}$ *for* $t \neq j$ *and* $i \neq j$.

Proof.

Lemma 12.3. $a_{i\ell}^{-1}\begin{bmatrix} k\ell & a_{kj} \\ a_{i\ell} & \underline{a_{ij}} \end{bmatrix} = -a_{k\ell}^{-1}\begin{bmatrix} a_{k\ell} & \underline{a_{kj}} \\ a_{i\ell} & a_{ij} \end{bmatrix}$.

Proof.

$$a_{i\ell}^{-1}(a_{ij} - a_{i\ell}a_{k\ell}^{-1}a_{kj}) = a_{i\ell}^{-1}a_{ij} - a_{k\ell}^{-1}a_{kj} = -a_{k\ell}^{-1}(a_{k\ell} - a_{k\ell}a_{i\ell}^{-1}a_{ij})$$

□

If something hods for 2×2 matrices, by the hereditary property it holds for all matrices. □

13 Lecture 12

Definition 13.1. Let $A = (a_{ij})$ is Cartier Foata if for all $i \neq k$ we have that $a_{ij}a_{k\ell} = a_{k\ell}a_{ij}$. □

Theorem 13.2. *For every matrix A that is Cartier-Foata and invertible then* $|A|_{ij} = (\det_{CF}(A^{ij}))^{-1} \cdot \det_{CF}(A)$ *where* $\det_{CF}(A) = \sum_{\sigma\in S_n}(-1)^{sgn(\sigma)}a_{\sigma(1)1}a_{\sigma(2)2}\cdots a_{\sigma(n)n}$.

This is equivalent to showing $\det_{CF}(A) = |A|_{11}|A^{11}|_{22}|A^{12,12}|_{33}\cdots a_{nn}$.

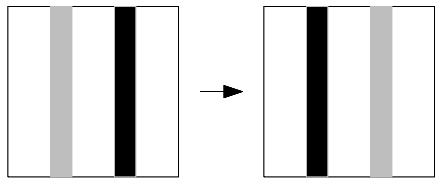
Lemma 13.3. *Let $B = (b_{ij})$ be Cartier Foata.*

(a) Let B' be the matrix obtained by switching columns i and j. Then $\det_{CF}(B) = -\det_{CF}(B')$.

(b) If a matrix B has two identical columns then by the above observation then $\det_{CF}(B) = 0$.

(c) $\det_{CF}(B) = \sum_{i=1}^{n}(-1)^{n+i}(\det_{CF} B^{in}) \cdot b_{in}$.

Proof. (a) follows since the entries in the same row commute. (b) follows from (a), and (c) also follows.



B $\qquad$ B'

$\square$

Proof.

$$(\det_{CF}(B^{11}), -\det_{CF}(B^{21}), \ldots, (-1)^{n-1}\det_{CF}(B^{n1}) \cdot B = (\cdots, \sum_{i=1}^{n}(-1)^i(\det_{CF} B^{i1}), \cdots) = \det_{CF}(B)(1,0,\ldots,0)B^{-1}$$
$$= (-\det_{CF}(B^{11}), -\det_{CF}(B^{21}), \ldots)$$

Then $(B^{-1})_{11} = (\det(B))^{-1}\det B^{11}$. $\square$

13.1 q-Cartier-Foata matrices

$a_{j\ell}a_{ik} = a_{ik}a_{j\ell}$ for all $i < j$ and $k < \ell$ and $a_{j\ell}a_{ik} = q^2 a_{ik}a_{j\ell}$ for $i < j$ and $k > \ell$, and $a_{jk}a_{ik} = qa_{ik}a_{jk}$ for all $i < j$.

Theorem 13.4. $\det_{q-CF}(A) = |A|_{11}|A^{11}|_{22}\cdots a_{nn}$ *and these product terms commute where*

$$\det_{q-CF}(B) = \sum_{\sigma\in S_n}(-q)^{inv(\sigma)} b_{\sigma(1)1}b_{\sigma(2)2}\cdots.$$

13.2 MacMahon's Master Theorem

Theorem 13.5 (MacMahon's Master Theorem)**.** *Let* $A = (a_{ij})_{n\times n}$, $a_{ij} \in \mathbb{C}$ *and define* $G(k_1, \ldots, k_n) = [x_1^{k_1}\cdots x_1^{k_n}]\prod(a_{i1}x_1 + \cdots + a_{in}x_n)^{k_i}$, *for integers* $(k_1, k_2, \ldots, k_n 0 \in \mathbb{N}^n$.

Let $T = diag(t_1, t_2, \ldots, t_n)$ *then*

$$\sum_{(k_1,k_2,\ldots,k_n)} G(k_1, k_2, \ldots, k_n)t_1^{k_1}\cdots t_n^{k_n} = \frac{1}{\det(I - TA)}$$

An application to this result is Dixon's identity.

Corollary 13.6.

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3 = (-1)^n\binom{3n}{n,n,n}$$

Proof. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. Then

$$G(2n,2n,2n) = [x_1^{2n}x_2^{2n}x_3^{2n}](x_2 - x_3)^{2n}(x_1 - x_2)^{2n}(x_3 - x_1)^{2n} = \sum_{k=0}^{2n}(-1)^{3k}\binom{2n}{k}.$$

MacMahon master theorem says that $G(2n, 2n, 2n)$ is also equal to

$$G(2n, 2n, 2n) = [t_1^{2n} t_2^{2n} t_3^{2n}] \frac{1}{\det(I - TA)},$$

where $I - TA = \begin{bmatrix} 1 & -t_1 & t_1 \\ t_2 & 1 & -t_2 \\ -t_3 & t_3 & 1 \end{bmatrix}$, and $\det(I - TA) = 1 + t_1t_2 + t_1t_3 + t_2t_3$. Thus

$$\begin{aligned} G(2n, 2n, 2n) &= [t_1^{2n} t_2^{2n} t_3^{2n}] \frac{1}{1 + (t_1t_2 + t_1t_3 + t_2t_3)} \\ &= [t_1^{2n} t_2^{2n} t_3^{2n}] \sum_{r=0}^{\infty} (-1)^r + (t_1t_2 + t_1t_3 + t_2t_3)^r \\ &= (-1)^{3n} [t_1^{2n} t_2^{2n} t_3^{2n}] (t_1t_2 + t_1t_3 + t_2t_3)^{3n} = (-1)^n \binom{3n}{n, n, n}. \end{aligned}$$

□

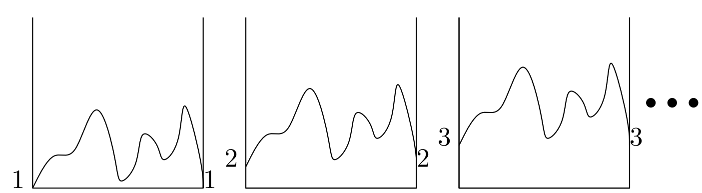
Proof MacMahon master theorem by Konvanlinka-Pak. Let $A = (a_{ij})$ where a_{ij} is in a ring and A is a Cartier-Foata matrix. Instead of proving the identity for complex numbers, we prove it for words. That is the MacMahon Master Theorem is equivalent to the following statement.

$$\sum_{(k_1, k_2, \ldots, k_n)} G(k_1, k_2, \ldots, k_n) = \frac{1}{\det_{CF}(I - A)}.$$

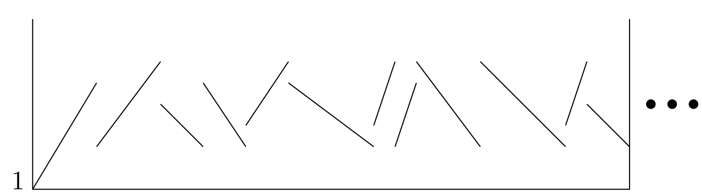
$G(k_1, k_2, \ldots, k_n)$ is a generating function of words (we can recover the t_i from the words). But

$$\begin{aligned} \frac{1}{\det_{CF}(I - A)} &= |I - A|_{11}^{-1} |I - A^{11}|_{22}^{-1} \cdots \\ &= \left(\frac{1}{I - A} \right)_{11} \left(\frac{1}{I - A^{11}} \right)_{22} \cdots \\ &= (I + A + A^2 + \cdots)_{11} (I - A^{11} - (A^{11})^2 + \cdots) \cdots \end{aligned}$$

In terms of words this is words that start and end in level i, but do not go below level i.



The meaning of $G(k_1, k_2, \ldots, k_n)$ we have paths where there are k_i steps leaving i and k_i steps going to i. The words in the LHS and RHS are the same.



□

14 Lecture 13: Heaps of pieces

The invetor of this theory was Xavier Viennot.

14.1 The Cartier-Foata monoid

Given a poset $P = \{[n], \preceq\}$ whose elements are $x_1, x_2, \ldots, x_n$. We look at words X under an equivalence $\sim$: $x_i x_j = x_j x_i$ if $i \not\preceq j$ and $j \not\preceq i$ i.e. if i and j are incomparable.

Let $F_P = \sum_{w \in X^*/\sim} w$, which is a noncommutative generating series.

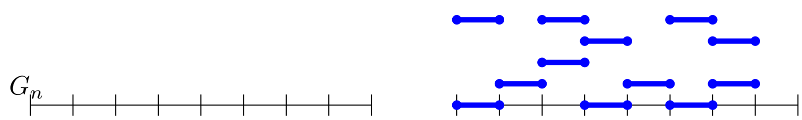
Then $\alpha : X \to R$ where $R = \mathbb{Z}[[q]]$ and where $\alpha(x_{i_1} x_{i_2} \cdots) = \alpha(x_{i_1})\alpha(x_{i_2}) \cdots$ and $\alpha(F) = \sum_{w \in X^*/\sim} \alpha(w)$ and $\alpha : x_i \to q$. So $\alpha(F_P) = \sum_{w \in X^*/\sim} q^{|w|}$.

14.2 Example Heaps of Pieces

Example 14.1. *Let $G = (V, E)$ be a finite graph.* Pieces *will be subgraphs of G. The poset P will be a finite collection of pieces. We say that $P_1 P_2 = P_2 P_1$ if $V(P_1) \cap V(P_2) = \varnothing$. We are interested in the words $P_1 P_3 P_4 P_1 P_1$ up to commutativity.*

Example 14.2. *Let G be the path with n vertices and the pieces P_i the edge from i to $i+1$. $F_H = \sum_{H \in \mathcal{H}} H$.*

The following is an example with 11 pieces for $n = 8$.



Theorem 14.3. *$\mathcal{F}_{\mathcal{H}} = \sum_{H \in \mathcal{H}} H$ where H stands for words in $\mathcal{P}$ the set of pieces. Then $F_{\mathcal{H}} = (\sum_{H \in \mathcal{D}} (-1)^{|H|} H)^{-1}$ where $\mathcal{D} \subset \mathcal{H}$ is the set of* simple pieces *D which consist of commuting pieces.*

Example 14.4. *In the previous example $\mathcal{D}$ consists of graphs of disjoint edges in from the path. So $F_n(q) = \sum_{H \in \mathcal{H}} q^{|H|} = \left(\sum_{H \in \mathcal{D}} (-q)^{|H|}\right)^{-1}$. Note that $A_1(q) = 1$, $A_2(q) = 1 - q$, $A_3(q) = 1 - 2q$, $A_4(q) = 1 - 3q + q^2$, $\ldots A_{n+1}(q) = A_n(q) + (-q)A_{n-1}(q)$. These are* Chebyshev polynomials *of the second kind, a family of Orthogonal polynomials. Thus $F_n(q)$ is $1/A_n(q)$. The case $n = 2$ gives $1 + x + x^2 + \cdots = 1/(1 - x)$.*

Theorem 14.3. We show with an involution that $(\sum_{H \in \mathcal{H}} H)(\sum_{D \in \mathcal{D}} D(-1)^{|D|} = 1$.

We do a linear order on P and always try to remove the smallest possible pieces in H and D. □

Open 14.5. *Find deterministic polynomial time algorithm to generate a matrix in $U_n(p)$ at distance $n^{1+\epsilon}$ from the identity.*

15 Lecture 14: Application heaps of pieces

Let $U(n, p)$ be the set of strict $n \times n$ upper triangular matrices with entries in $\mathbb{F}_p$. Let $S = \{E_k(a)\}$ where $(E_k(a))_{ij} = \begin{cases} 1 & \text{if } i = j \neq k \\ a & \text{if } i = k, j = k+1 \\ 0 & \text{otherwise} \end{cases}$. These matrices $E_i(a)$ are called *elementary transvections.*

Question 15.1. *What is the diameter of* $\Gamma(U(n,p),S)$*?*

Recall that the given a finitely generated group G, and $S \subseteq G$ where $S = S^{-1}$ and $< S >= G$. The *Cayley graph* $\Gamma(G,S)$ is the graph with vertices $g \in G$ and edges (g, gs) and (gs, g) where $g \in G$ and $s \in S$.

Proposition 15.2. *In all situations* $\Gamma(G,S) > \lfloor \log_{|S|} |G| \rfloor - 1$

Proof. Distance 0 we have one element (the identity), distance 1 we have $|S|$ elements, ..., distance k we have $|S|^k$ elements. If $|S|^k > |G|$ then $k < \operatorname{diam}\Gamma$. Then using sums of geometric series $|G| < 1 + m + m^2 + \cdots + m^k = \frac{m^{k+1}}{m-1} < m^{k+1}$ so $k+1 > \log m^{|G|}$. ☐

Corollary 15.3. $d(n,p) \geq \log_{(n-1)(p-1)} p^{\binom{n}{2}}$.

Proof.

$$= \frac{\log p^{\binom{n}{2}}}{\log(n) + \log(p)} = \frac{\binom{n}{2}(\log p)}{\log(n) + \log(p)}.$$

If p is fixed and $n \to \infty$ then we get $d(n,p) > C(p)\frac{n^2}{\log n}$. ☐

Theorem 15.4 (Ellenberg)**.** *For p fixed then* $d(n,p) = \Theta(n^2)$*.*

Lemma 15.5. $d(n,p) = O(n^2)$.

Example 15.6. *Given the first row it takes $O(n)$ to clear the first row. This is because if we look at $a_{12} = 1$ then we can take the next row. Let t_n be the number of steps to get any first row from any matrix. Then $t_n = t_{n-1} + 1$.*

Remark 15.7. If we look at this problem over the reals then the lower bound is also quadratic (by the dimension of upper triangular matrices). The hard part is to show the lower bound when p is small, like 2. ☐

Lemma 15.8. $d(n,p) = \Omega(n^2)$.

Proof. If we show that $\sum_{w \in S^*, \ell(w) \leq \ell} \#$ distinct words $< |G|$ then diam $> \ell$. These are heaps of intervals $(i, i+1)$. So

$$\sum_{w \in S^*, \ell(w) \leq \ell} \# \text{ distinct words } < \sum_{H, |H| \leq \ell} (p-1)^{|H|} < |G|$$

But we know that

$$\sum (p-1)^{|H|} = \frac{1}{A_{n-1}(p-1)},$$

where $A_n(x)$ are the Chebyshev polynomials of the second kind. In general

$$A_n(x) = \frac{1}{2^{n+2}y}\left((1+y)^{n+2} - (1-y)^{n+2}\right),$$

where $y = \sqrt{1-4x}$. If $x < 1/4$ then $A_n(x)$ has no roots. So using a bit of calculus shows that for all p

$$\sum (p-1)^{|H|} = \frac{1}{A_{n-1}(p-1)} \leq (4(p-1))^{\ell}.$$

Therefore $d(n,p) \geq \log_{4p} p^{\binom{n}{2}} = C \cdot n^2$.

☐

16 Lecture 15:

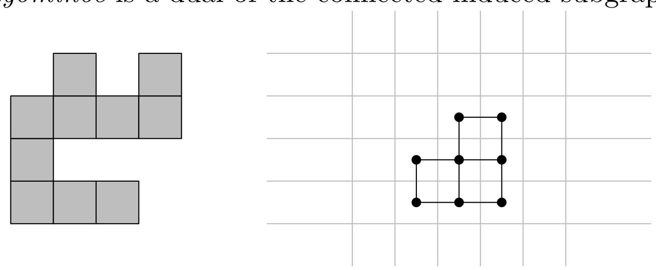
We had success estimating the diameter of $U(n,q)$. What if we try other groups. One bad example is the symmetric group. $\mathrm{diam}(S_n, \{(i,i+1) \mid i = 1, \dots, n\}$

Proposition 16.1. diam $> \log_{n+1} n! = \frac{\log e^{n \log n}}{\log n} \sim n$.

The theorem implies in this case that the diameter $cn \log n$. But a combintorial argument gives that $dist_T(1,\sigma) = inv(\sigma)$ so diam $= \binom{n}{2} = \Theta(n^2)$.

To use heaps of pieces the Cayley graph has to be special.

Today we will see another application of heaps of pieces to count *polyominoes* or *animals*. A *polyominoe* is a dual of the connected induced subgraph of $\mathbb{Z}^2$.

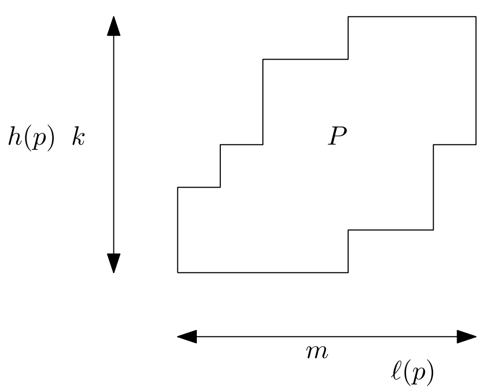


Conjecture 16.2. *If $a(n)$ is the number of polyominoes of area n then $a(n) \sim Cn^{\alpha}\lambda^n$ some $\lambda > 1$.*

The number of such polyominoes is in the sequence [**?**, A000105].

16.1 Parallelogram polyominoes

Exercise 16.3. *Let $pp(n,k)$ be the number of parallelogram polyominoes with dimensions n and k. This number is $\frac{1}{n+k}\binom{n+k}{k}\binom{n+k}{k+1}$, the Narayana numbers. Note that $\sum_{n+k=m} p(n,k) = C_m$, the mth Catalan number.*



Given a parallologram polyominoe P, let $h(P)$ be the height of P, $\underline{\ }(P)$ the width of P, and $a(P)$ the area of P.

Let $F(x,y,q) = \sum_P x^{b(P)} y^{h(P)} q^{a(P)}$ then

Theorem 16.4. *$F(x,y,q) = yA(x,y,q)/B(x,y,q)$.*

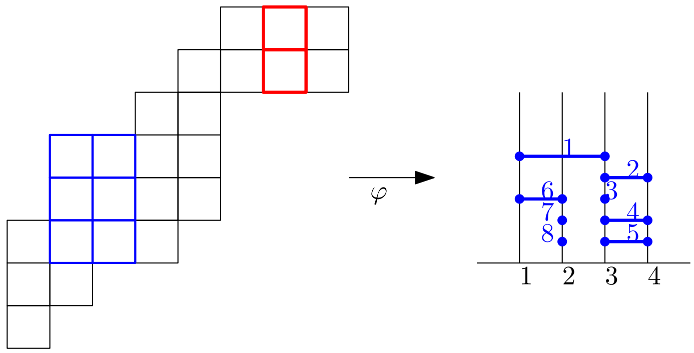
16.1.1 Plan of proof

1. we will relate parallelogram polyominoes ot heaps, 2. generalize the main theorem of heaps to accommodate for restricting to heaps that have restricted maxima, 3. complete the proof of Theorem 16.4.

Theorem 16.5. *Let $\mathcal{M} \subseteq \mathcal{P}$ then*

$$\sum_{H \in \mathcal{H}, \max(H) \subseteq \mathcal{M}} H = \left(\sum_{H \in \mathcal{D}} (-1)^{|H|} H \right)^{-1} \left(\sum_{H \in \mathcal{D}_{\mathcal{P} \setminus M}} (-1)^{|H|} H \right).$$

The proof of this generlization is very similar to the proof of Theorem 14.3.
Given a parallelogram polyominoe



We record the *column heights* $(3, 4, 3, 4, 4, 2, 2, 2)$ and *intersection lengths* $(a_1, a_2, \ldots, a_n)$ where $a_1 = 1$ is $(1, 3, 3, 3, 3, 1, 2, 2, 1)$

The *heaps* are intervals $[a_n, c_n] \circ [a_{n-1}, c_{n-1}] \circ \cdots \circ [a_1, c_1]$. The pieces are $[i, j]$ for $i \leq j$.

Example 16.6. *In* $s_1 = [1, 3], s_2 = [3, 4], s_3 = [3, 3], s_4 = [3, 4], s_5 = [3, 4], s_6 = [1, 2], s_7 = [2, 2], s_8 = [2, 2]$.

Lemma 16.7. *There is a correspondence between parallelogram polyominoes and heaps such that $b(p)$ goes to $H(P)$ and $h(P)$ goes to the sum of the lengths and $a(P)$ becomes the sum of c_is.*

17 Lecture 16

$F(x, y, q) = \sum_p x^{b(P)} y^{h(P)} q^{a(P)}$, where $b(P)$ is the width of the heap, $h(P)$ is the height of the heap, $a(P)$ is the area of P.

Theorem 17.1.

$$F(x, y, q) = \frac{q \sum_{n=0}^{\infty} (-1)^n x^{n+1} q^{\binom{n+2}{3}} / (q, q)_n (yq, q)_{n+1}}{\sum_{n=0}^{\infty} (-1)^n x^n / (q, q)_n (yq, q)_n},$$

where $(\alpha, q)_n = (1 - \alpha)(1 - \alpha q) \cdots (1 - \alpha q^{n-1})$

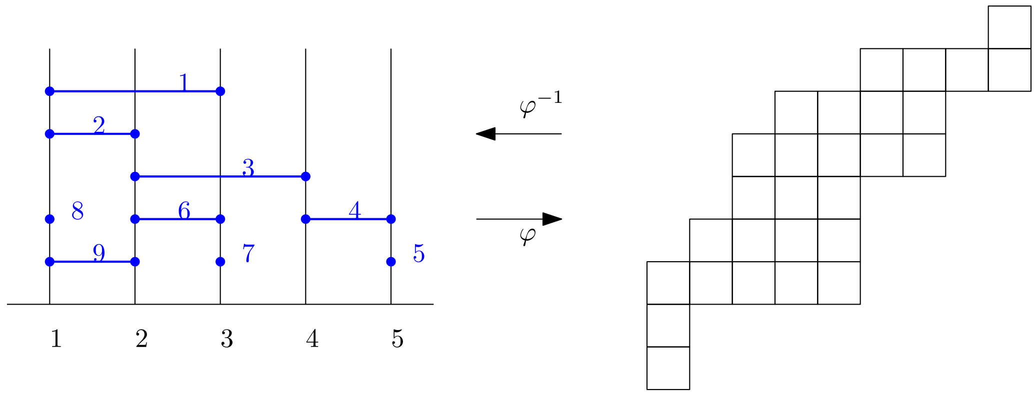
As a corollary we get

Corollary 17.2 (Bousquet-Mélou and Viennot)**.**

$$F(x, y, q) = \cfrac{xyq}{1 - q(x + y) - \cfrac{xyq^3}{1 - q^3(x+y) - \cfrac{xyq^5}{1 - q^5(x+y) - xyq^7}}}.$$

Lemma 17.3. *The parallelogram polyominoes (PP) are in bijection with heaps on $\mathcal{P} = \{[a_i, c_i]\}$ where a_i are the column intersections and c_i are the column lengths.*

We give an example of the reverse bijection.



Lemma 17.4.

$$\sum_{H \in \mathbb{R}_\Omega, \max(H) \subseteq \mathcal{M}} H = (\sum_{H \in \mathcal{D}_\Gamma} (-1)^{|H|} H)(\sum_{H \in D_{\Gamma \setminus M}} (-1)^{|H|} H).$$

Let $\mathcal{D}_{\mathcal{P}_n} =$ 

.

Lemma 17.5. *$\mathcal{D}_{\mathcal{P}_n}$ are in bijection with pars of partitions winto parts $\leq n$ which add up to $a(P)$.*

Proof. $1^{m_1} 2^{m_2} 3^{m_3} \cdots \to \lambda$ and $1^{k_1} 2^{k_2} 3^{k_3} \cdots \to \mu$ where $m_1 = \alpha_{n-1} + 1$ and $k_1 = \beta_{n-i+1}$. $\square$

Next note that

$$\frac{1}{(1-q)} \frac{1}{(1-q^2)} \cdots \frac{1}{(1-q^n)} = \sum_{\lambda : \lambda_1 \leq n} q^{|\lambda|}.$$

and

$$\frac{1}{(1-yq)} \frac{1}{(1-yq^2)} \cdots \frac{1}{(1-yq^n)} = \sum_{\lambda : \lambda_1 \leq n} y^{b(\lambda)} q^{|\lambda|}.$$

Consider the tridiagonal matrix

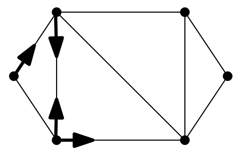
$$A = \begin{bmatrix} \alpha_1 & 1 & & \\ -1 & \alpha_2 & 1 & 0 \\ & -1 & \alpha_3 & 1 \\ \vdots & & & \vdots \\ & & -1 & \alpha_n \end{bmatrix}$$

then $|A|_{11} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{\ddots}{\alpha_n}$.
If we want to compute A^{-1}, let B be such that $A = I - B$ then

$$I + B + B^2 + B^3 + \cdots$$

18 Lecture 17: Heaps and cycles and more

Definition 18.1. $G = (V, E)$ be an undirected graph. $\mathcal{P}_G$ is the set of pieces $\{(v, e) \mid e = \overrightarrow{(v, v')}\}$ where (v, e) and (v, e') are comparable. $\square$



$\mathcal{H}_{\mathcal{P}} = \mathcal{H}_G$ is the set of heaps H.

Definition 18.2. We say that H is **balanced** if for all v the $\#(v,e) \in H$ equals the number of $(w, \overrightarrow{(w,v)})$ in H □

Theorem 18.3.

$$\sum_{H \in \mathcal{H}_G,\ balanced} H = \sum_{K \in \mathcal{C}_G} K,$$

where $\mathcal{C}_G$ is the set of heaps of (simple) cycles in G.

This result is useful since the RHS allows us to get rid of the balanced condition.

By Theorem 14.3 we have that

$$\sum_{K \in \mathcal{C}_G} K = \left(\sum_{K \text{ simple heaps of simple cycles}} (-1)^K K)^{-1} \frac{1}{\sum \det(I - M_G)(-1)^{|K|}} \right),$$

where $M_G = (m_{ij})$ is the adjacency matrix of G (note that $m_{ij} = 1$ iff there is an edge $\overrightarrow{(v_i, e)}$ where $e = (v_i, v_j)$.

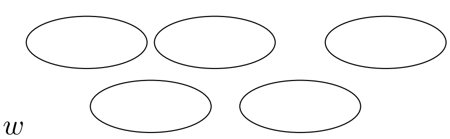
By the MacMahon Master Theorem (Theorem 13.5) we get that

18.1 Pyramid

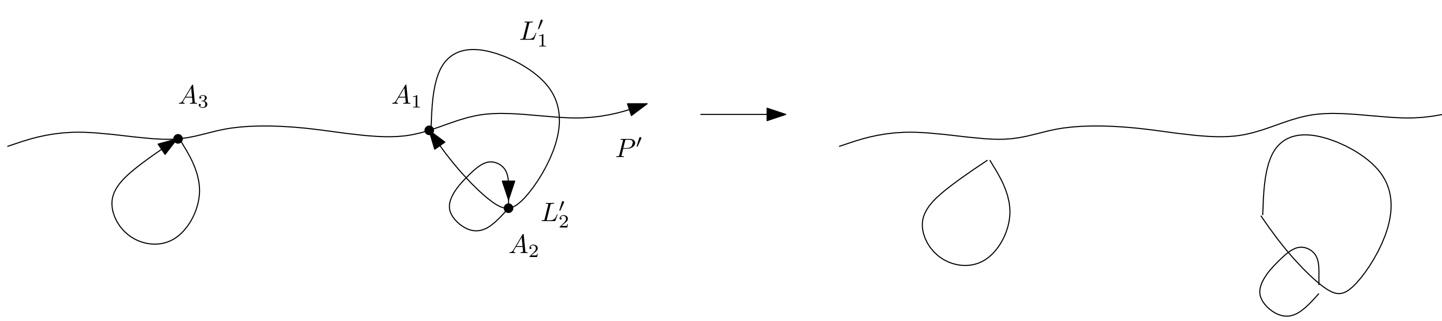
Theorem 18.4.

$$\sum_{paths\ P: v \to w} P = \sum_{L \in \mathcal{H}_G} r$$

self-avoiding walks $v \to w$



Proof. We traverse the path and as we encounter loops we separate them from the path and continue. The resulting configuration is a self avoiding walk and a heap of cycles.



□

18.2 Wilson's algorithm

We start with a graph G

1. Run LERW from vertex 1 to the vertex R.
2. Pick the minimal unvisited vertex x to path P_1.
3. This stops with a spanning tree.

Theorem 18.5 (D. Wilson 1997)**.** *This produces a uniform random spanning tree.*

Proof. The first step is to give an analogue algorithm to produce a uniform random tree. The algorithm is as follows. For each vertex not in R create a pice uniformly.

$$\sum \text{words} = (\sum_T T) = (\sum_{H \text{ heaps of cycles}} H).$$

□

Acknowledgements: Stephen DeSalvo for providing notes and clarifications.

19 Lecture 18: The BEST theorem

The name of the theorem comes from De **B**ruijn, Aurdenne-**E**hrenfest, **S**mith and **T**utte.

Definition 19.1 (deBruijn sequence)**.** $B(n)$ a sequence that contains every 0-1 word of length n cyclically exactly once. □

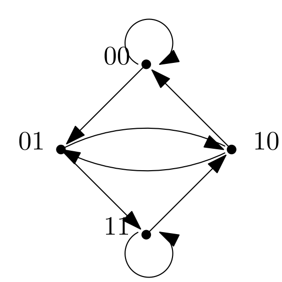
For example, for $n = 3$, 00010111.

Proposition 19.2. *Let $B(n)$ be the number of deBruijn sequence of cyclic length 2^n. Then $B(n) \geq 1$.*

Theorem 19.3. $B(n) = 2^{2^{n-1}-n}$

For some history of the problem, de Riviere introduced the problem in 1864 and conjectured a formula and Saite-Marie proved the theorem. This was forgotten and Posthumus conjectured it again in 1944 and de Bruijn proved it in 1946.

Proof. We build a directed graph $\mathcal{B}(n)$ with vertices $V = \{0,1\}^{n-1}$ and arcs $(x_1, x_2, \ldots, x_{n-1}) \rightarrow$



$(x_2, x_3, \ldots, x_n)$. Then Eulerian circuits in $\mathcal{B}(n)$ corresponds to deBruijn n-sequences.

By the following theorem, we know that $\mathcal{B}(n)$ has an Eulerian tour.

Theorem 19.4 (Euler, August 26, 1735)**.** *A connected directed graph G has an Eulerian cycle if and only if $indeg(v) = outdeg(v)$ for all vertices v in $V(G)$.*

For $\mathcal{B}(n)$ we have that $outdeg(v) = 2$, so $\mathcal{B}(n)$ has an Eulerian cycle. By the next result that was proved 200 years later allows us to compute the number of Eulerian cycles of a graph.

Theorem 19.5 (BEST)**.** *Fix v_0 in V then the number $\mathcal{E}(G)$ of Eulerian cycles of a directed graph G is*

$$\mathcal{E}(G) = \#\textit{directed trees rooted at } v_0 \prod_{v \in V} (outdeg(v) - 1)!.$$

Proof. $\prod_{v \in V, v \neq v_0} outdeg(v)!(outdeg(v_0) - 1)!$ are the possible orderings of the outdegrees. If for each vertex $v \in V$ $v \neq v_0$ we record the last outgoing edge from v, these set of edges form a directed tree oriented towards v_0. So

$$\prod_{v \in V} (outdeg(v) - 1)! \cdot \#\text{directed trees rooted at } v_0$$

counts Eulerian circuits. □

□

Remark 19.6. Up until the year 2000 we did not know the number of Eulerian cycles in K_{2n-1} and $K_{2n,2n}$. In this case the BEST theorem does not apply because the graphs are not directed. □

20 Lecture

21 Lecture

22 Lecture

Theorem 22.1. *The following are equivalent:*

1. *polynomino tilings*
2. *Wang tilings*
3. *walks in graphs*
4. *$\mathcal{F}$ $\mathbb{N}$-rational functions.*

Theorem 22.2 (Berstel-Soittola)**.** *If $F(t) = \frac{P(t)}{Q(t)}$ then $F \in \mathcal{F}$ if and only if*

0. *$F \in \mathbb{N}[[t]]$*
1. *If $\rho := \min_{z \text{ pole of } F} |z|$ then ρ is a pole.*
2. *If $|z| = \rho$ pole of F then $z^k = \rho^k$.*

Theorem 22.3 (Berstel)**.** *Suppose $F(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_n > 0$ and $F(t)$ is $\mathbb{R}_+$-rational, i.e. in $\mathcal{R}$ then*

0. *$\alpha, t \in \mathcal{R}$, $\alpha \geq 0$*
1. *$F, b \in \mathcal{R}$ then $F + b, F \cdot b \in \mathcal{R}$,*
2. *$[1]F = 0$ for $F \in \mathcal{R}$ and $\frac{1}{1-F}$ is in $\mathcal{R}$.*

Definition 22.4. Let $\mathcal{R}$ be the minimal class of generating functions satisfying the three conditions above. $\square$

Theorem 22.5 (Berstel)**.** *Suppose $F(t) = \sum_{n=0}^{\infty} a_n t^n$ with $a_n > 0$ and $F(t)$ is $\mathbb{R}_+$-rational (in $\mathcal{R}$) then we have conditions 1. and condition 2. from Berstel-Soittola's theorem.*

As a corollary of Theorems 22.3 and 22.5,

Corollary 22.6. *For $F \in \mathcal{R} \cap \mathbb{N}[[t]]$*

Lemma 22.7. *$F = \sum_{n=0}^{\infty} g_n t^n$ then $F = P(t)/Q(t)$, $\rho = \min_{z \in \Pi_F} |z|$ where Π_F is the set of poles of F then $\rho \in \Pi_F$.*

Proof. Let $z \in \mathbb{C}$ and $|z| < \rho$ then

$$|F(z)| = |\sum_n a_n z^n| \leq \sum_n a_n |z|^n = F(|z|).$$

Take $z_0 \in \Pi_F$ with $|z_0| = \rho$ with multiplicity m then $H(z) = (z_0 - z)^m F(z)$ where $H(z)$ is analytic and $H(z_0) \neq 0$. If ρ is a pole with multiplicity ,m then $G(z) = (\rho - z)^m F(z)$, $G(\rho) = 0$ and $\lim_{r \to 1, r<1}($
$rho - \rho r)^m F(\rho \cdot r) = 0$.

If $z = r z_0$ then $\lim_{r \to 1, r<1} z_0^m (1-r)^m F(r z_0) > 0$. So $\lim_{r \to 1, r<1} \rho^m (1-r)^m F(r\rho) > 0$. This is a contradiction to the previous equality. $\square$

Lemma 22.8. *$F = \sum a_n t^n$, $G = \sum b_n t^n$ and ρ_F, ρ_G are as above then $\rho_{F+G} = \min\{\rho_F, \rho_G\}$ and $\rho_{F \cdot G} = \min\{\rho_F, \rho_G\}$.*

Theorem 22.5. We need to check that condition 2 in Berstel-Soittola holds under the operation $1/(1-F)$. $\square$

Example 22.9.

$$F(t) = \frac{1}{1-(3t)^n} - \frac{1}{1-(2t)^n}$$

By Theorem 22.2 then $F(t) \in \mathcal{F}$.

23 Lecture

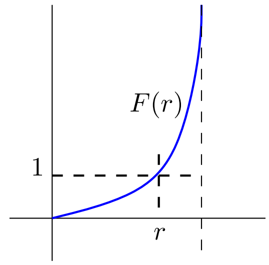
We continue proving Berstel's theorem.

Theorem 23.1 (Berstel)**.** *Let $F(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_n \geq 0$. Suppose $F(t)$ is $\mathbb{R}_+$-rational. Let $\rho_F = \min_{\rho \in \Pi_F} |\rho|$ where Π_F is the set of the poles of F. (1) Then ρ_F is also a pole of F and (2) every pole ρ of F such that $|\rho| = |\rho_F|$ satisfies $\rho^k = \rho_F^k$ for some k.*

Proof. We have checked (1). For (2) we have checked that $F, G \in \mathcal{R}$ then $F + G, F \cdot G \in \mathcal{R}$. If $[1]F = 0$ then $1/(1-F) \in \mathcal{R}$ and $\rho_{F+G} = \min(\rho_F, \rho_G)$.

The poles of $F^* = 1/(1-F)$ are the zeroes of $1 - F$. $F(\rho_F) = \sum_{n=0}^{\infty} a_n \sigma_F^n = \infty$ by definition. Now $F(r)$, by definition $F(0) = 0$, and $F(r)$ for $r \in [0, \rho_F)$ is increasing and $\lim_{r \to \rho_F} F(r) = \infty$

Then there exists r, $0 < r < \rho_F$ such that $F(r) = 1$. Then r is a pole of F^*. Let z be a pole of F^*, $|z| \leq r$.



Claim: $(z/r)^k = 1$ for some k.

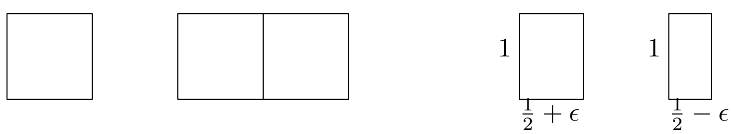
This is because $1 = \sum_{n=0}^{\infty} a_n z^n = Re(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n Re(z^n) \leq \sum_{n=0}^{\infty} a_n |z|^n \leq \sum_{n=0}^{\infty} a_n r^n = F(r) = 1$. So $F(z) = F(r)$ and therefore, $a_n Re(z^n) = a_n r^n$ for all n. Since $F(t) \neq 0$ then $a_k \neq 0$ for some k. Thus $Re(z^k) = r^k$. This proves (2). $\square$

23.1 Irrational Tilings

Let T be a set of irrational tiles of height 1. $a_n(T)$ be the number of ways to tile the strip $1 \times n$ with the tiles from T. We are interested in studying the generating function $\sum_{n=0}^{\infty} a_n t^n$.

Example 23.2.

1. *If the tiles are the two tiles in the right of the figure we get $a_n(T) = F_n$.*
2. *If the tiles are the two tiles on the right of the figure where ϵ is irrational we get $a_n(T) = \binom{2n}{n}$ but $\sum_{n=0}^{\infty} \binom{2n}{n} x^n = 1/\sqrt{1-4t}$.*



Theorem 23.3 (Garrabrant-Pak)**.** *For all T irrational tiles $A(t) = \sum_{n=0}^{\infty} a_n(T) t^n$ then $A(t) = Diag F(x_1, x_2, \ldots, x_k)$, i.e. $a_n = [x_1^n x_2^n \cdots x_k^n] F$ where F is a $\mathbb{N}$-rational function of k variables.*

Definition 23.4. The class $\mathcal{F}_k$ of $\mathbb{N}$-rational functions in k variables is the class satisfying:

0. $0, x_1, \ldots, x_k \in \mathcal{F}_k$,
1. $F, G \in \mathcal{F}_k$ then $F + G \in \mathcal{F}_k$,
2. If $[1]F = 0$ then $\frac{1}{1-F} \in \mathcal{F}_k$

$\square$

23.2 Asymptotics of classes of generating functions

1. (Rational) $F(t) = P(t)/Q(t)$ for some $P, Q \in \mathbb{Z}[t]$, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r}$.

   We use the following asymptotic notation $a_n \propto C \cdot n^{\alpha} \cdot \lambda^n$ where $C \in \mathbb{A}, \alpha \in \mathbb{Z}, \lambda \in \mathbb{A}$. This means that a_n is asymptotically a finite sum of multiples of such powers.

   Example 23.5. *Let $a_n = 2^n + (-2)^n$, then $a_n \propto 2^n$.*

2. ($\mathbb{N}$-rational) Let $F \in \mathcal{F}$ be $\mathbb{N}$-rational. There exists m such that $c_n \sim C_j n^{\lambda_j} \lambda_j^n$ for $n = j$ (mod m).

3. (Algebraic) $p_0(t)F^k + p_1(t)F^{k-1} + \cdots + p_k(t) = 0$ where $p_i(t)$ are polynomials.

   Example 23.6. *If $a_n = \binom{2n}{2n}$ then $A(t) = \sum_{n=0}^{\infty} = 1/\sqrt{1-4t}$. This is algebraic.*

   If $F = 1 + tF^2$, then $F(t) = \sum_{n=0}^{\infty} C_n t^n$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ are the Catalan numbers.

   Theorem 23.7. $a_n \propto C n^{\alpha} \lambda^n e^{r(n)} (n!)^{\sigma} (logn)^{\gamma}$

4. (D-finite) $c_0(n)a_n = c_1(n)a_{n-1} + c_2(n)a_{n-2} + \cdots + c_r(n)a_{n-r}$, where $c_i(n)$ are polynomials in n over $\mathbb{Z}$.

   Claim: Theorem 23.7 for algebraic functions works for D-finite.

   Example 23.8. *Let $a_n = \frac{1}{n!}$, then $A(t) = e^t$ which is not algebraic but $A(t)$ is D-finite.*

   Example 23.9. *Let a_n be the number of permutations of n that are involutions. Then $a_n = a_{n-1} + (n-1)a_{n-2}$. The exponential generating function of a_n is $\sum_{n=0}^{\infty} \frac{a_n}{t}{}^n/n! = e^{t-t^2/2}$. This is not algebraic but it is D-finite.*

Clearly $1 \supset 2$ (proper) and $1 \subset 3 subset 4$ both proper.

Theorem 23.10. *Suppose $F = DiagP(x_1, x_2, \ldots, x_n)/Q(x_1, x_2, \ldots, x_n)$ then F is D-finite.*

Moreover

Theorem 23.11. *If $F = DiagP/Q$ where P, Q is D-finite then F is D-finite.*

Now the class 5. of generating series of irrational tilings are contained in 4.

Theorem 23.12 (Fustenberg)**.** *Every F algebraic is $F = DiagP(x,y)/Q(x,y)$ and $DiagP(x,y)/Q(x,y)$ is algebraic.*

Example 23.13. *$C_n = [x^n y^n](1 - x/y)/(1 - x - y)$ since $[x^n y^n]1/(1 - x - y) = \binom{2n}{n}$ and $[x^{n+1}y^{n-1}]1/(1 - x - y) = \binom{2n}{n-1}$ and $C_n = \binom{2n}{n} - \binom{2n}{n-1}$. Also Rowland and Yassawi showed that*

$$C_n = [x^n y^n] \frac{y(1 - 2xy - 2xy^{@})}{1 - x - 2xy - xy^2}.$$

So Fustenberg's theorem does not necessarily give a unique expression as diagonals.